# MFin Econometrics I

## *Session 3: t-distribution, Simple Linear Regression, OLS assumptions and properties of OLS estimators*

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# $t$-distribution

## What if $\sigma_Y$ is un-known? (almost always)

- Recall: $\bar{Y} \sim N(\mu_Y, \frac{\sigma_Y{}^2}{n})$ and $\frac{\bar{Y}-\mu_Y}{\frac{\sigma_Y}{\sqrt{n}}} \sim N(0,1)$
  - If $(Y_1, ..., Y_n)$ are i.i.d. (and $0 < \sigma_Y^2 < \infty$) then, when $n$ is large, the distribution of $\bar{Y}$ is well approximated by a normal distribution: Why? *CLT*
  - If $(Y_1, ..., Y_n)$ are independently and identically drawn from a Normal distribution, then *for any value of $n$*, the distribution of $\bar{Y}$ is normal: Why? *Sums of normal r.v.s are normal*
- But we almost never know $\sigma_Y$. We use sample standard deviation ($s_Y$) to estimate the unknown $\sigma_Y$
- Consequence of using the sample s.d. in the place of the population s.d. is an increase in uncertainty. *Why?*
  - $s_Y/\sqrt{n}$ is the standard error of the mean : estimate from sample, of st. dev. of sample mean, over all possible samples of size $n$ drawn from the population

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# $t$-distribution

## Using $s_Y$: "estimator" of $\sigma_Y$

- ${s_Y}^2 = \frac{1}{\text{sample size-1}} \sum_{i=1}^{\text{sample size}} (Y_i - \bar{Y})^2$
  - Why sample size - 1 in this estimator?
  - Degrees of freedom (d.f.): the number of *independent observations* available for estimation. Sample size less the number of (linear) parameters estimated from the sample
- Use of an *estimator* (i.e., a random variable) for $\sigma_Y$ (and thus, the use of an estimator for $\sigma_{\bar{Y}}$) motivates use of the fatter-tailed $t$-distribution in the place of $N(0,1)$
- The law of large numbers (LLN) applies to ${s_Y}^2$ : ${s_Y}^2$ is, in fact, a "sample average" (How?)
- LLN: If sampling is such that $(Y_1, ..., Y_n)$ are i.i.d., (and if $E(Y^4) < \infty$), then ${s_Y}^2$ *converges in probability* to ${\sigma_Y}^2$:
  - ${s_Y}^2$ is an average, not of $Y_i$, but of its square (hence we need $E(Y^4) < \infty$)

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# $t$-distribution

## Aside: $t$-distribution probability density

$$f(t) = \frac{\Gamma(\frac{d.f.+1}{2})}{\sqrt{d.f.\pi}\Gamma(\frac{d.f.}{2})}(1 + \frac{t^2}{d.f.})^{-\frac{d.f.+1}{2}}$$

- $d.f.$ : Degrees of freedom: the only parameter of the $t$-distribution;
  - $\Gamma()$ (Gamma function): $\Gamma(z) = \int_0^\infty t^{z-1}e^{-t}dt$
- Mean: $E(t) = 0$
- Variance: $V(t) = d.f./(d.f. - 2)$ for $d.f. > 2 \quad \geq 1$, converges to 1 as $d.f.$ increases (Compare: $N(0,1)$)

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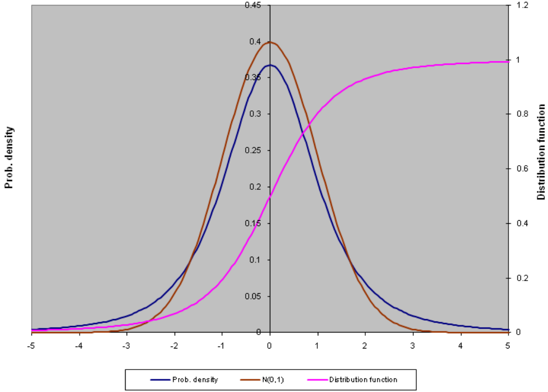
# $t$-distribution

## $t$-distribution pdf and CDF: d.f.=3



# $t$-distribution

## $t$ - statistic

$$t_{d.f.} = \frac{\bar{Y} - \mu}{s_Y / \sqrt{n}} \qquad d.f. = n - 1$$

- test statistic for the sample average same form as $z$-statistic: Mean 0, Variance $\rightarrow 1$
- If the r.v. $Y$ is is normally distributed in population, the test-statistic above for $\bar{Y}$ is $t$-distributed with $d.f. = n - 1$ degrees of freedom

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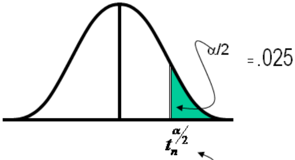
# $t$-distribution

## $t$ - distribution, table and use




| Degrees of Freedom | $t_{.100}$ | $t_{.05}$ | $t_{.025}$ | $t_{.01}$ | $t_{.005}$ |
|---|---|---|---|---|---|
| 1 | 3.078 | 6.314 | 12.706 | 31.821 | 63.657 |
| 2 | 1.886 | 2.92 | 4.303 | 6.965 | 9.925 |
| . | . | . | . | . | . |
| . | . | . | . | . | . |
| 20 | 1.325 | 1.725 | 2.086 | 2.528 | 2.845 |
| . | . | . | . | . | . |
| . | . | . | . | . | . |
| 200 | 1.286 | 1.653 | 1.972 | 2.345 | 2.601 |
| ∞ | 1.282 | 1.645 | 1.96 | 2.326 | 2.576 |

# $t$-distribution

## $t \to$ SND

- If d.f. is moderate or large ($> 30$, say) differences between the $t$-distribution and $N(0,1)$ *critical values* are negligible
- Some 5% critical values for 2-sided tests:

| degree of freedom | 5% $t$-distribution critical value |
|---|---|
| 10 | 2.23 |
| 20 | 2.09 |
| 30 | 2.04 |
| 60 | 2.00 |
| $\infty$ | **1.96** |

# $t$-distribution

## Comments on $t$ distribution, contd.

- The $t$ distribution is only relevant when the sample size is small. But then, for the $t$ distribution to be correct, the populn. distrib. of $Y$ must be Normal Why?
  - The popln r.v. ($\bar{Y}$) must be Normally distributed for the test-statistic (with $\sigma_Y$ estimated by $s_Y$) to be $t$-distributed
  - If sample size small, CLT does not apply; the popln distrib. of $Y$ must be Normal for the popln. distrib. of $\bar{Y}$ to be Normal (sum of Normal r.v.s is Normal)
- So if sample size small and $\sigma_Y$ estimated by $s_Y$, then test-statistic to be $t$-distributed if popln. is Normally distributed
- If sample size large (e.g., $> 30$), the popln. distrib. of $\bar{Y}$ is Normal irrespective of distrib. of $Y$ - CLT (we saw that as d.f. increases, distrib. of $t_{d.f.}$ converges to $N(0, 1)$)
- Finance / Management data: Normality assumption dubious. e.g., earnings, firm sizes etc.

# $t$-distribution

## Comments on $t$ distribution, contd.

- So, with large samples,
- or, with small samples, but with $\sigma$ known, and a normal population distribution

$$
\begin{aligned}
Pr(-z_{\alpha/2} \leq \frac{\bar{Y}-\mu}{\sigma/\sqrt{n}} \leq z_{\alpha/2}) &= 1-\alpha \\
Pr(\bar{Y} - z_{\alpha/2}\frac{\sigma}{\sqrt{n}} \leq \mu \leq \bar{Y} + z_{\alpha/2}\frac{\sigma}{\sqrt{n}}) &= 1-\alpha \\
Pr(\bar{Y} - 1.96\frac{\sigma}{\sqrt{n}} \leq \mu \leq \bar{Y} + 1.96\frac{\sigma}{\sqrt{n}}) &= 0.95
\end{aligned}
$$

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# $t$-distribution

## Comments on $t$ distribution, contd.

- With small samples ($<30$ d.f.),
- drawn from an approximately normal population distribution,
- with the unknown $\sigma$ estimated with $s_Y$,
- for test statistic $\frac{\bar{Y}-\mu}{s_Y/\sqrt{n}}$:

$$Pr(\bar{Y} - t_{d.f.,\alpha/2}\frac{s_Y}{\sqrt{n}} \leq \mu \leq \bar{Y} + t_{d.f.,\alpha/2}\frac{s_Y}{\sqrt{n}}) = 1-\alpha$$

Q: What is $t_{d.f.,\alpha/2}$?

# Simple linear regression model

## Population Linear Regression Model: Bivariate

$$Y = \beta_0 + \beta_1 X + u$$

- Interested in conditional probability distribution of $Y$ given $X$
- Theory / Model : the conditional mean of $Y$ given the value of $X$ is a linear function of $X$
  - $X$ : the independent variable or regressor
  - $Y$ : the dependent variable
  - $\beta_0$ : intercept
  - $\beta_1$ : slope
  - $u$ : regression disturbance, which consists of effects of factors other than $X$ that influence $Y$, as also *measurement errors* in $Y$
  - $n$ : sample size, $\quad Y_i = \beta_0 + \beta_1 X_i + u_i \quad i = 1, \cdots, n$

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# Simple linear regression model

## Linear Regression Model - Issues

- Issues in estimation and inference for linear regression estimates are, at a general level, the same as the issues for the sample mean
- Regression coefficient is just a glorified mean
- Estimation questions:
  - How can we estimate our model? i.e., "draw" a line/plane/surface through the data ?
  - Advantages / disadvantages of different methods ?
- Hypothesis testing questions:
  - How do we test the hypothesis that the population parameters are *zero*?
    - Why test if they are zero?
  - How can we construct confidence intervals for the parameters?

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# Simple linear regression model

## Regression Coefficients of the Simple Linear Regression Model

- How can we estimate $\beta_0$ and $\beta_1$ from data?
  - $\bar{Y}$ is the ordinary least squares (OLS) estimator of $\mu_Y$:
  - Can show that $\bar{Y}$ solves:

$$min_X \sum_{i=1}^{n} (Y_i - X)^2$$

- By analogy, OLS estimators of the unknown parameters $\beta_0$ and $\beta_1$, solve:

$$min_{\hat{\beta}_0,\, \hat{\beta}_1} \sum_{i=1}^{n} (Y_i - (\hat{\beta}_0 + \hat{\beta}_1 X_i))^2$$

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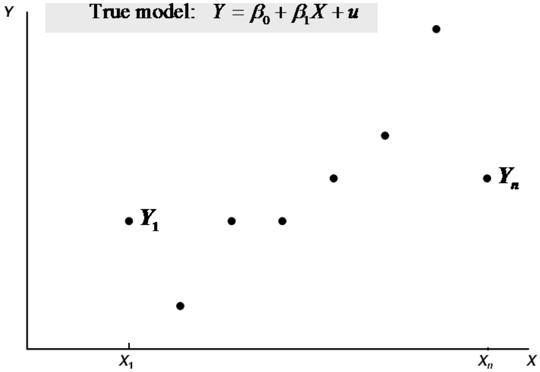
# Simple linear regression model

## OLS in pictures (1)

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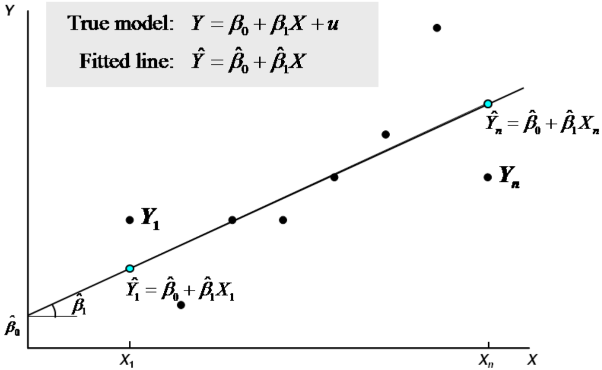
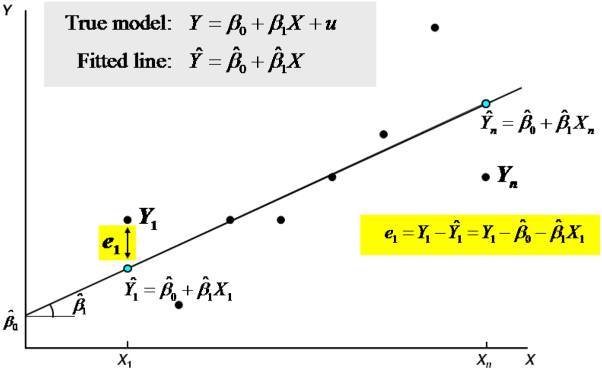
# Simple linear regression model

## OLS in pictures (2)



# Simple linear regression model

## OLS in pictures (3)

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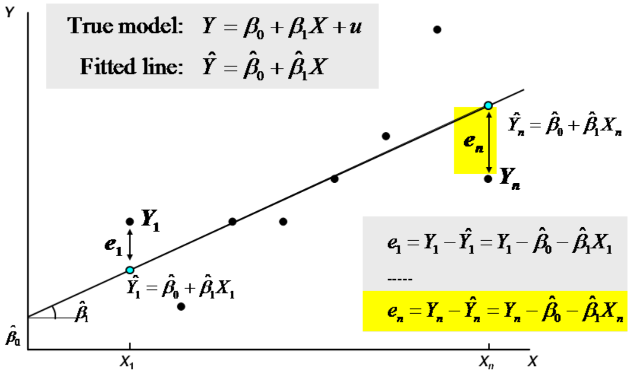
# Simple linear regression model

## OLS in pictures (4)

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# Simple linear regression model

## OLS method of obtaining regression coefficients

- The sum : $e_1^2 + .. + e_n^2$, is the Residual Sum of Squares (RSS), a measure of total *error*
  - RSS is a function of both $\hat{\beta}_0$ and $\hat{\beta}_1$ (How?)

$$\begin{aligned} \text{RSS} &= (Y_1 - \hat{\beta}_0 - \hat{\beta}_1 X_1)^2 + .. + (Y_n - \hat{\beta}_0 - \hat{\beta}_1 X_n)^2 \\ &= \sum_{i=1}^{n} (Y_i - \hat{\beta}_0 - \hat{\beta}_1 X_i)^2 \end{aligned}$$

- Idea: Find values of $\hat{\beta}_0$ and $\hat{\beta}_1$ that minimise RSS
- $\frac{\partial RSS(\cdot)}{\partial \hat{\beta}_0} = -2 \sum (Y_i - \hat{\beta}_0 - \hat{\beta}_1 X_i) = 0$
- $\frac{\partial RSS(\cdot)}{\partial \hat{\beta}_1} = -2 \sum X_i (Y_i - \hat{\beta}_0 - \hat{\beta}_1 X_i) = 0$

# Simple linear regression model

## Ordinary Least Squares, contd.

- $\frac{\partial RSS}{\partial \hat{\beta}_0} = 0$ and $\frac{\partial RSS}{\partial \hat{\beta}_1} = 0$
- Solving these two equations together:
- $\hat{\beta}_1 = \frac{\frac{1}{n}\sum(X_i - \bar{X})(Y_i - \bar{Y})}{\frac{1}{n}\sum(X_i - \bar{X})^2} = \frac{\sum(X_i - \bar{X})(Y_i - \bar{Y})}{\sum(X_i - \bar{X})^2} = \frac{Cov(X,Y)}{Var(X)}$
- $\hat{\beta}_0 = \bar{Y} - \hat{\beta}_1 \bar{X}$

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# Simple linear regression model

**Linear Regression Model: Interpretation**

- Exercise

# Linear regression model: Evaluation

## Measures of Fit (i): Standard Error of the Regression (SER) and Root Mean Squared Error (RMSE)

- Standard error of the regression (SER) is an estimate of the dispersion (st.dev.) of the distribution of the disturbance term, $u$;
- Equivalently, of $Y$, conditional on $X$
- How close are $Y$ values to $\hat{Y}$ values? can develop confidence intervals around any prediction
  - $SER = \sqrt{\frac{1}{n-2}\sum_{i=1}^{n}(e_i - \bar{e})^2} = \sqrt{\frac{1}{n-2}\sum_{i=1}^{n} {e_i}^2}$
  - SER converges to root mean squared error (RMSE)
  - $RMSE = \sqrt{\frac{1}{n}\sum_{i=1}^{n}(e_i - \bar{e})^2} = \sqrt{\frac{1}{n}\sum_{i=1}^{n} e_i^2}$
  - RMSE denominator has $n$ SER has $(n-2)$
  - Why?

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# Linear regression model: Evaluation

## Measures of Fit (ii): $R^2$

- How much of the variance in $Y$ can we explain with our model?
- Without the model, the best estimate of $Y_i$ is the sample mean $\bar{Y}$
- With the model, the best estimate of $Y_i$ is conditional on $X_i$ and is the fitted value $\hat{Y}_i = \hat{\beta}_0 + \hat{\beta}_1 X_i$
- How much does the error in estimate of $Y$ reduce with the model?

# Linear regression model: Evaluation

## Goodness of fit

- The model: $Y_i = \hat{Y}_i + e_i$

$$
\begin{aligned}
Var(Y) &= Var(\hat{Y} + e) \\
&= Var(\hat{Y}) + Var(e) + 2Cov(\hat{Y}, e) \\
&\overset{\text{Why?}}{=} Var(\hat{Y}) + Var(e) \\
\frac{1}{n}\sum(Y - \bar{Y})^2 &= \frac{1}{n}\sum(\hat{Y} - \bar{\hat{Y}})^2 + \frac{1}{n}\sum(e - \bar{e})^2 \\
\sum(Y - \bar{Y})^2 &= \sum(\hat{Y} - \bar{\hat{Y}})^2 + \sum(e - \bar{e})^2
\end{aligned}
$$

- Total Sum of Squares (TSS) = Explained Sum of Squares (ESS) + Residual Sum of Squares (RSS)

$$
R^2 = \frac{ESS}{TSS} = \frac{\sum(\hat{Y}_i - \bar{Y})^2}{\sum(Y_i - \bar{Y})^2}
$$

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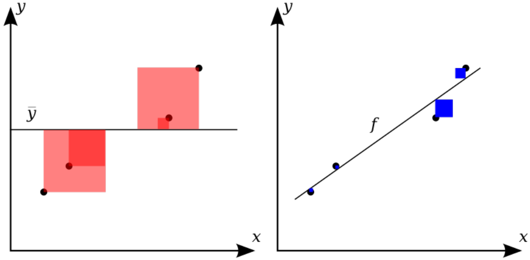
# Linear regression model: Evaluation

## Goodness of fit

- $TSS = ESS + RSS$
- $R^2 = \frac{ESS}{TSS} = \frac{\sum(\hat{Y}_i - \bar{Y})^2}{\sum(Y_i - \bar{Y})^2} = 1 - \frac{\sum e_i^2}{\sum(Y_i - \bar{Y})^2}$
- $\sqrt{R^2} = \frac{Cov(Y, \hat{Y})}{st.dev.(Y) st.dev.(\hat{Y})} = r_{Y,\hat{Y}}$



# Properties of OLS Estimators

**After regression estimates**

- In practical terms, we wish to:
  - quantify sampling uncertainty associated with $\hat{\beta}_1$
  - use $\hat{\beta}_1$ to test hypotheses such as $\beta_1 = 0$
  - construct confidence intervals for $\beta_1$
- all these require knowledge of the sampling distribution of the OLS estimators (based on the probability framework of regression)

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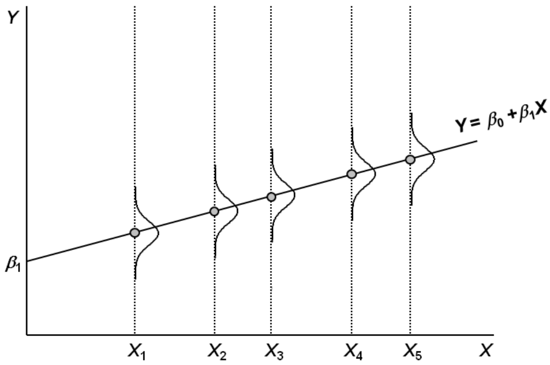
# Properties of OLS Estimators

## Properties of Estimators and the Least Squares Assumptions

- What kinds of estimators would we like?
- unbiased, efficient, consistent
- Under what conditions can these properties be guaranteed?
- We focus on the sampling distribution of $\hat{\beta}_1$ (Why not $\hat{\beta}_0$?)
  - The results below do hold for the sampling distribution of $\hat{\beta}_0$ too.

# Properties of OLS Estimators

## Assumption 1: $E(u|X = x) = 0$



# Properties of OLS Estimators

**Assumption 1: $E(u|X = x) = 0$**

- Conditional on $X$, $u$ does not tend to influence $Y$ either positively or negatively.
- Implication : Either $X$ is not random, or,
- If $X$ is random, it is distributed independently of the disturbance term, $u$: $Cov(X, u) = 0$
- This will be true if there are no relevant omitted variables in the regression model (i.e., those that are correlated to $X$)

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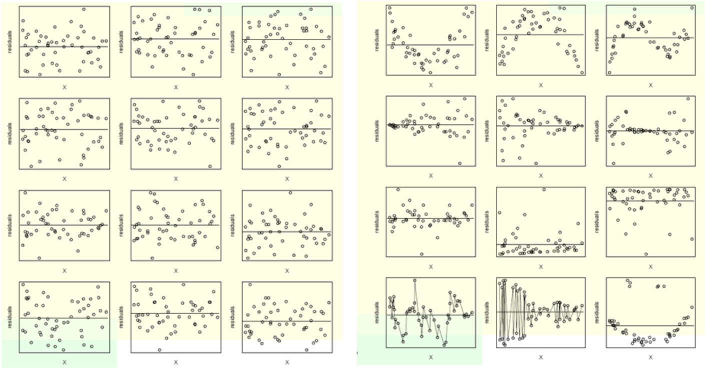
# Properties of OLS Estimators

## Assumption 1: Residual plots that pass and that fail



# Properties of OLS Estimators

**Aside: include a constant in the regression**

- Suppose $E(u_i) = \mu_u \neq 0$
- Suppose $u_i = \mu_u + v_i$ where $v_i \sim N(0, {\sigma_V}^2)$
- Then $Y_i = \beta_0 + \beta_1 X_i + v_i + \mu_u = (\beta_0 + \mu_u) + \beta_1 X_i + v_i$

# Properties of OLS Estimators

**Assumption 2: $(X_i, Y_i)$, $i = 1, ..., n$ are i.i.d.**

- This arises naturally with simple random sampling procedure
- Because most estimators are linear functions of observations,
- Independence between observations helps in obtaining the sampling distributions of the estimators

# Properties of OLS Estimators

**Assumption 3: Large outliers are rare**

- A large outlier is an *extreme* value of $X$ or $Y$
- Technically, $E(X^4) < \infty$ and $E(Y^4) < \infty$
  - Note: If $X$ and $Y$ are bounded, then they have finite fourth moments (income, etc.)
- Rationale : a large outlier can influence the results significantly

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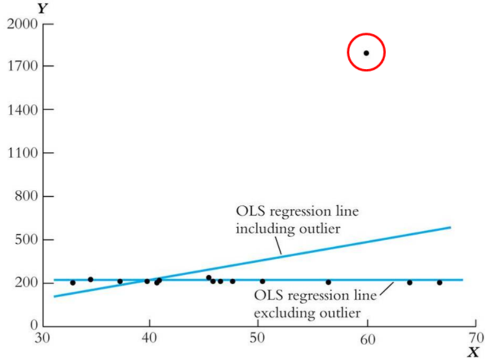
# Properties of OLS Estimators

## Assumption 3: OLS sensitive to outliers

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# Properties of OLS Estimators

## Sampling distribution of $\hat{\beta}_1$

- If the three Least Squares Assumptions (mean zero disturbances, i.i.d. sampling, no large outliers) hold,
- then the exact (finite sample) sampling distribution of $\hat{\beta}_1$ is such that:
  - $\hat{\beta}_1$ is unbiased, that is, $E(\hat{\beta}_1) = \beta_1$
  - $Var(\hat{\beta}_1)$ can be determined
    - Other than its mean and variance, the exact distribution of $\hat{\beta}_1$ is complicated and depends on the distribution of $(X, u)$
  - $\hat{\beta}_1$ is consistent: $\hat{\beta}_1 \to_p \beta_1$ $plim(\hat{\beta}_1) = \beta_1$
  - So when $n$ is large, $\frac{\hat{\beta}_1 - \beta_1}{\sqrt{Var(\hat{\beta}_1)}} \sim N(0, 1)$ (by CLT)
- This parallels the sampling distribution of $\bar{Y}$

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# Properties of OLS Estimators

## Mean of the sampling distribution of $\hat{\beta}_1$

- $Y = \beta_0 + \beta_1 X + u$
- unbiasedness

$$\begin{aligned}
\hat{\beta}_1 &= \frac{Cov(X,Y)}{Var(X)} = \frac{Cov(X,[\beta_0 + \beta_1 X + u])}{Var(X)} \\
&= \frac{Cov(X,\beta_0) + Cov(X,\beta_1 X) + Cov(X,u)}{Var(X)} \\
&= \frac{0 + \beta_1 Cov(X,X) + Cov(X,u)}{Var(X)} \\
&= \beta_1 + \frac{Cov(X,u)}{Var(X)}
\end{aligned}$$

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# Properties of OLS Estimators

## Unbiasedness of $\hat{\beta}_1$

- $\hat{\beta}_1 = \frac{Cov(X,Y)}{Var(X)} = \beta_1 + \frac{Cov(X,u)}{Var(X)}$
- To investigate unbiasedness, take Expectation
- $E(\hat{\beta}_1) = \beta_1 + \frac{1}{Var(X)} E(Cov(X,u)) = \beta_1$
    - Expected value of $Cov(X,u)$ is zero (Why?)
- $\hat{\beta}_1$ is an unbiased estimator of $\beta_1$

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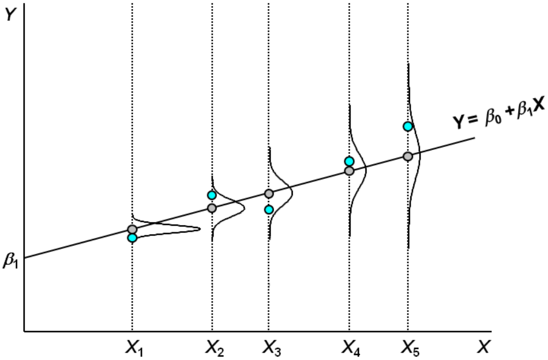
# Linear regression model: further assumptions

## Homoscedasticity and Normality of residuals

- $u$ is homoscedastic
- $u \sim N(0, {\sigma_u}^2)$
- These assumptions are more restrictive
- However, if these assumptions are not violated, then other desirable properties obtain

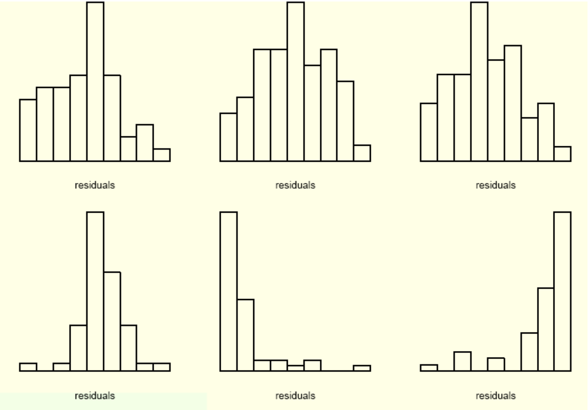
# Linear regression model: further assumptions

## Heteroscedasticity: one example



# Linear regression model: further assumptions

## Normality of disturbances: histograms of residuals that 'pass' and 'fail'



# Linear regression model: further assumptions

## Normality of disturbances 2: q-q plots of residuals that ‘pass’ and ‘fail’