MFin Econometrics I

Session 4: Precision of OLS estimators, Multiple regression models, Multicollinearity, F-tests for goodness of fit

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Gauss-Markov Theorem

Efficiency of OLS - The Gauss-Markov Theorem

- OLS estimators $\hat{\beta}_0$ and $\hat{\beta}_1$ are linear functions
  - of $Y_1, ..., Y_n$, in bivariate regression
- Under assumptions 1-4 (mean zero conditional distributions of disturbances, i.i.d. sampling, no outliers, homoscedasticity):
- the OLS estimators have the *smallest variance* among all linear estimators (i.e., of all estimators that are linear functions of $Y_1, ..., Y_n$)
  - Aside: proof available in standard texts (if you are interested)

Gauss-Markov Theorem

Efficiency of OLS esimators (2)

- Under all five assumptions - i.e., including normally distributed errors:
- OLS estimators have the smallest variance among all consistent estimators (i.e., linear or nonlinear functions of $Y_1, ..., Y_n$)
  - Aside: proof available in standard texts
- This is a strong result: OLS is a better choice than any other consistent estimator
- An estimator that is not consistent is a very poor choice, so OLS really is the best we can do - *if all five assumptions hold*

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Gauss-Markov Theorem

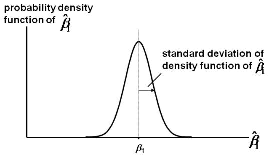
Limitations of OLS

- OLS is more sensitive to outliers than some other estimators
- Recall that to estimate the population mean, if there are outliers, then the sample median is preferred to the sample mean
    - the median is less sensitive to outliers - it has smaller variance than OLS (mean) when there are outliers
- Similarly, in regression, if there are outliers, then there are other estimators that are more efficient (have smaller variances)
    - Q: What are outliers? How can we treat them?
    - Aside: Robust statistics
- All said, OLS is the most popular estimator in applied regression analysis

Precision of OLS estimators

Variance of OLS estimators under homoscedasticity

- Simple linear regression model: $Y = \beta_0 + \beta_1 X + u$
- Focus on the slope coefficient : more "interesting" and useful (why?). All arguments apply to the intercept as well
- Variances (of the sampling distributions) of regression coefficients (under homoscedasticity)
- $\sigma_{\hat{\beta}_1}{}^2 = \frac{\sigma_u{}^2}{nVar(X)}$
  - $\sigma_{\hat{\beta}_0}{}^2 = \frac{\sigma_u{}^2}{n}\{1 + \frac{\bar{X}^2}{Var(X)}\}$



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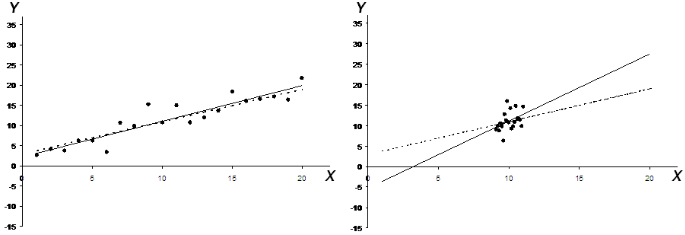
Precision of OLS estimators

Variance of OLS estimators under homoscedasticity (2)

- Larger $V(X)$ (and larger n), lower is V(OLS estimators)



Precision of OLS estimators

Variance of OLS estimators under heteroscedasticity

- ${\sigma_{\hat{\beta}_1}}^2 = \frac{Var[(X_i - E(X))u_i]}{n[Var(X)]^2}$
  - Aside: derived in textbooks
- For comparison, Recall, for i.i.d. random variable Y:
  - $E(\bar{Y}) = \mu_Y \qquad Var(\bar{Y}) = \frac{{\sigma_Y}^2}{n}$

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Precision of OLS estimators

So, some dispersions we are concerned with

- Simple linear regression model: $Y = \beta_0 + \beta_1 X + u$
- ${\sigma_{\hat{\beta}_1}}^2 = \frac{{\sigma_u}^2}{nVar(X)}$ (homoscedastic case)
- ${\sigma_u}^2$ estimated with ${s_u}^2 = \frac{n}{n-2}Var(e)$ (Why the $\frac{n}{n-2}$ term?)
  - $E[Var(e)] = \frac{n-2}{n}{\sigma_u}^2$ (proof available in textbooks)
  - So if we define ${s_u}^2 = \frac{n}{n-2}Var(e)$, then $E[{s_u}^2] = E[\frac{n}{n-2}Var(e)] = {\sigma_u}^2$: unbiased
- $s.e.(\hat{\beta}_1) = \sqrt{\frac{{s_u}^2}{nVar(X)}} = \sqrt{\frac{Var(e)}{(n-2)Var(X)}}$
- $s.e.(\hat{\beta}_0) = \sqrt{\frac{{s_u}^2}{n}\left(1 + \frac{\bar{X}^2}{Var(X)}\right)} = \sqrt{\frac{Var(e)}{n-2}\left(1 + \frac{\bar{X}^2}{Var(X)}\right)}$

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Hypothesis testing on regression coefficients

Null and Alternate hypotheses

- Objective: test a hypothesis, e.g., $\beta_1 = \beta_1^*$, and reach a probabilistic conclusion whether this hypothesis is correct or incorrect, relative to an alternative
- General setup
- Null hypothesis and two-sided alternative:
  - $H_0 : \beta_1 = \beta_1^* \; Vs. \; H_a : \beta_1 \neq \beta_1^*$
- Null hypothesis and one-sided alternative:
  - $H_0 : \beta_1 = \beta_1^* \; Vs. \; H_a : \beta_1 > \beta_1^*$
  - $H_0 : \beta_1 = \beta_1^* \; Vs. \; H_a : \beta_1 < \beta_1^*$

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Hypothesis testing on regression coefficients

Approach

- General approach: construct test-statistic, and compute p-value (or compare with critical value from t or $N(0,1)$)
- In general: test statistic $= \frac{\text{estimator - hypothesised value}}{\text{std. error of estimator}}$
- to test β_1^*: $t = \frac{\hat{\beta}_1 - \beta_1^*}{s.e.(\hat{\beta}_1)}$
  - Recall: $s.e.(\hat{\beta}_1) =$ the square root of the estimator of the variance of the sampling distribution of $\hat{\beta}_1$
- For $H_0 : \beta_1 = \beta_1^* \; Vs. \; H_a : \beta_1 \neq \beta_1^*$: Reject at 5% significance level if $|t| > 1.96$ (if large sample)
- Q: What is a confidence interval for β_1?

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Multiple Regression estimators

Multiple regression with two explanatory variables: example

- A model for automobile sales : where registrations depend on “list price” and “rebate”
- Registrations $= \beta_0 + \beta_1 Price + \beta_2 Rebate + u$
- Or a model for MPO1 marks : where marks depend on “individual work” and “group work”
  Marks $= \beta_0 + \beta_1 \text{IW} + \beta_2 \text{GW} + u$

Multiple Regression estimators

Multiple regression with two explanatory variables: example

- Population regression function

$$Y_i = \beta_0 + \beta_1 X_{1i} + \beta_2 X_{2i} + u_i$$

- Sample regression function

$$\hat{Y}_i = \hat{\beta}_0 + \hat{\beta}_1 X_{1i} + \hat{\beta}_2 X_{2i}$$

- Residual

$$e_i = Y_i - \hat{Y}_i = Y_i - \hat{\beta}_0 - \hat{\beta}_1 X_{1i} - \hat{\beta}_2 X_{2i}$$

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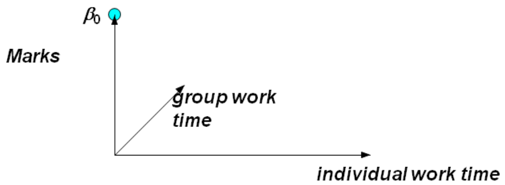
Multiple Regression estimators

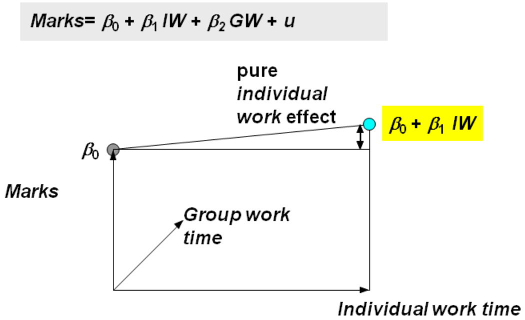
Multiple regression in pictures (1)

Marks= $\beta_0 + \beta_1 IW + \beta_2 GW + u$



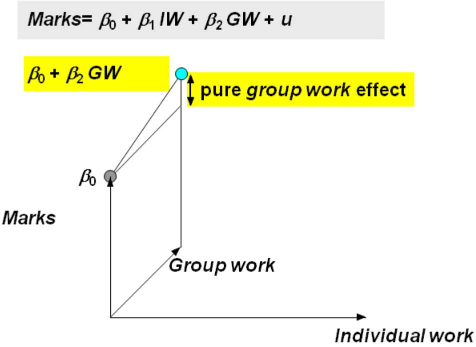
Multiple Regression estimators

Multiple regression in pictures (2)



Multiple Regression estimators

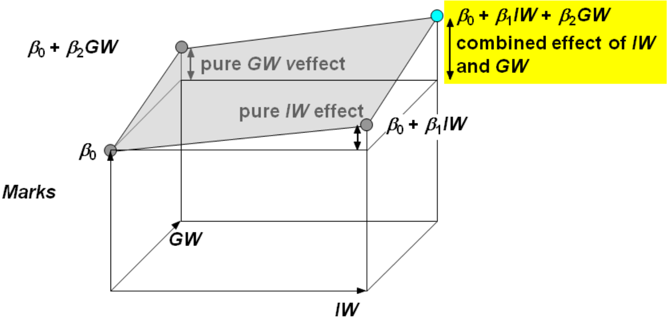
Multiple regression in pictures (3)



Multiple Regression estimators

Multiple regression in pictures (4)

$$Marks = \beta_0 + \beta_1 IW + \beta_2 GW + u$$



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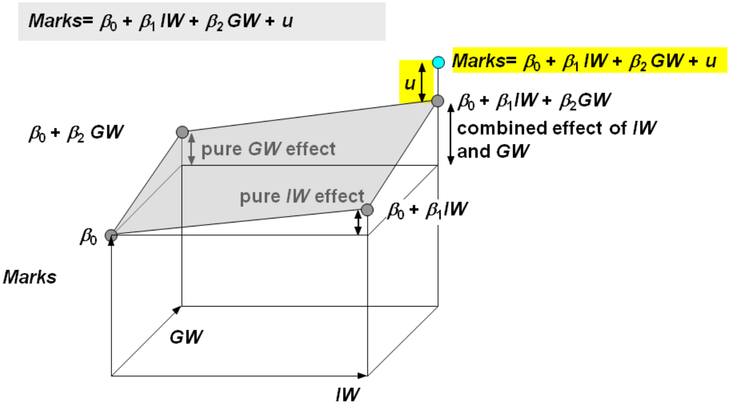
Multiple Regression estimators

Multiple regression in pictures (5)



Multiple Regression estimators

Multiple regression with two explanatory variables: example

- Residual sum of squares in terms of unknown estimators

$$RSS = \sum e_i^2 = \sum (Y_i - \hat{Y}_i) = \sum (Y_i - \hat{\beta}_0 - \hat{\beta}_1 X_{1i} - \hat{\beta}_2 X_{2i})^2$$

- Minimising error

$$\frac{\partial\, RSS}{\partial \hat{\beta}_0} = 0; \quad \frac{\partial\, RSS}{\partial \hat{\beta}_1} = 0; \quad \frac{\partial\, RSS}{\partial \hat{\beta}_2} = 0$$

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Multiple Regression estimators

Multiple regression with two explanatory variables: example

- $\hat{\beta}_0 = \bar{Y} - \hat{\beta}_1 X_{1i} - \hat{\beta}_2 X_{2i}$
- $\hat{\beta}_1 = \frac{Cov(X_1,Y)Var(X_2) - Cov(X_2,Y)Cov(X_1,X_2)}{Var(X_1)Var(X_2) - [Cov(X_1,X_2)]^2}$
- $\hat{\beta}_2 = \frac{Cov(X_2,Y)Var(X_1) - Cov(X_1,Y)Cov(X_1,X_2)}{Var(X_1)Var(X_2) - [Cov(X_1,X_2)]^2}$
- Derivations in textbooks, if interested
- Expressions simpler for general models with matrix notation

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R^2 and adjusted R^2

R^2: Coefficient of determination

- $\sum(Y_i - \bar{Y})^2$ is the total sum of squares (TSS)
- $\sum(\hat{Y}_i - \bar{Y})^2$ is the explained sum of squares (ESS)
- $\sum e_i^2$ is the residual sum of squares (RSS)
- TSS=ESS+RSS

$$R^2 = \frac{ESS}{TSS} = \frac{\sum(\hat{Y}_i - \bar{Y})^2}{\sum(Y_i - \bar{Y})^2} = 1 - \frac{\sum e_i^2}{\sum(Y_i - \bar{Y})^2}$$

- R^2 always increases with the number of regressors.
- Cannot compare ‘larger’ and ‘smaller’ models with this measure of “fit”

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R^2 and adjusted R^2

Adjusted R^2

- "Adjusted R^2": makes comparison possible by "penalising" inclusion of more regressors

$$\bar{R}^2 = 1 - \frac{n-1}{n-K}\frac{RSS}{TSS}$$

- $\bar{R}^2$ *can* fall when unrelated regressors are included
- $\bar{R}^2 < R^2$
- If n large, the two *can* be close
- Can R^2 ever be negative?
  - Yes, if the best-fit model is inappropriate and fits the data worse than a horizontal line at the mean Y value

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Multiple Regression estimators: properties

Multiple regression estimators: desirable properties

- If the model is correctly specified, and the "Gauss Markov Assumptions" are not violated, OLS estimators of the multiple regression model coefficients ($\hat{\beta}_k$) are:
  - Unbiased
  - Efficient
  - Consistent

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Multiple Regression estimators: properties

Precision of Multiple regression estimators

- $Y_i = \beta_0 + \beta_1 X_{1i} + \beta_2 X_{2i} + u_i$
- Population variance of a slope coefficient, say $\hat{\beta}_1$:
  $\sigma_{\hat{\beta}_1}{}^2 = \frac{\sigma_u{}^2}{nVar(X_1)} \times \frac{1}{1-r_{X_1,X_2}{}^2}$
  - Recall: in the simple linear model $Y_i = \beta_0 + \beta_1 X_i + u_i$
    $\sigma_{\hat{\beta}_1}{}^2 = \frac{\sigma_u{}^2}{nVar(X)}$
    - $E[Var(e)] = \frac{n-2}{n}\sigma_u{}^2$
    - $s_u{}^2 = \frac{n}{n-2}Var(e)$
    - $\sigma_{\hat{\beta}_1}{}^2 = \frac{\frac{n}{n-2}Var(e)}{nVar(X)}$
- Sample estimate of the variance of a slope coefficient, $\hat{\beta}_1$ for $Y_i = \beta_0 + \beta_1 X_{1i} + \beta_2 X_{2i} + u_i$:
- $s.e.(\hat{\beta}_1)^2 = \frac{s_u{}^2}{nVar(X_1)} \times \frac{1}{1-r_{X_1,X_2}{}^2} = \frac{Var(e)}{(n-3)Var(X_1)} \times \frac{1}{1-r_{X_1,X_2}{}^2}$

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Multicollinearity

What is Multicollinearity

- Situation when two or more predictor variables are highly (linearly) correlated
  - i.e., $\beta_0 + \beta_1 X_{1i} + \beta_2 X_{2i} + \cdots + \beta_K X_{Ki} + v_i = 0$; Variance of v_i is small
  - Some of the β_ks may be zero in the above
- Multicollinearity does not reduce predictive power or reliability of the *model as a whole*
- But reduces precision of estimators relating to individual predictors (why?)

Multicollinearity

Diagnosing Multicollinearity

- $Y_i = \beta_0 + \beta_1 X_{1i} + \cdots + \beta_K X_{Ki} + u_i$
- Population variance of the OLS estimator for a typical regression coefficient, e.g., $\hat{\beta}_k$:
- ${\sigma_{\hat{\beta}_k}}^2 = \frac{{\sigma_u}^2}{nVar(X_k)} \times \frac{1}{1-{R_k}^2}$
- ${R_k}^2$ is the R^2 for the regression of X_k against all other explanatory variables in the model:
  - $X_k = \gamma_0 + \gamma_1 X_1 + \cdots + \gamma_{k-1} X_{k-1} + \gamma_{k+1} X_{k+1} + \cdots + \gamma_K X_K + \nu_i$
- If there is no linear relation between X_k and the other explanatory variables in the model, ${R_k}^2 \approx 0$
- Diagnostic for multicollinearity is *related to* ${R_k}^2$

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Multicollinearity

Variance inflation factor

- Variance Inflation Factor$_k = \frac{1}{1-{R_k}^2}$
- Assesses the degree to which variance (s.e. of the coefficient) is inflated because regressor k is not orthogonal to the other regressors
- However, the sampling distribution of VIF is not known
- Rule of thumb: Consider multicollinearity a significant problem if average $VIF > 1$ or individual $VIF > 10$ (for any regressor)

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Multicollinearity

Alleviating Multicollinearity

- $Y_i = \beta_0 + \beta_1 X_1 + \beta_2 X_2 + u_i$
- ${\sigma_{\hat{\beta}_1}}^2 = \frac{{\sigma_u}^2}{nVar(X_1)} \times \frac{1}{1 - {r_{X_1,X_2}}^2}$
  - Reduce σ_u^2 by including further relevant variables in the model
  - Increase the number of observations, n
  - Increase $Var(X_.)$
  - Reduce r_{X_1,X_2}
  - Combine the correlated variables
  - Drop some of the correlated variables

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χ^2 and F Distributions

Chi-squared Distribution ${\chi_K}^2$

- If $Y_i \sim N(0,1)$, then
- $\sum_{i=1}^{K} {Y_i}^2 \sim {\chi_K}^2$ distribution, with K degrees of freedom

$$pdf: \; f(y,K) = \begin{cases} \frac{1}{2^{K/2}\Gamma(K/2)} y^{(K/2)-1} e^{-y/2} & for\ y > 0 \\ 0 & for\ y \leq 0 \end{cases}$$

  - $\Gamma(\cdot)$ is the Gamma function
- $E(\sum_{i=1}^{K} {Y_i}^2) = K$

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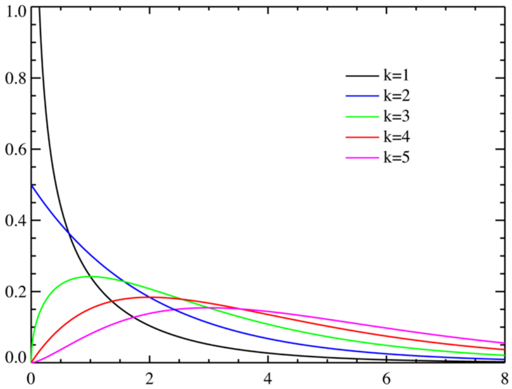
χ^2 and F Distributions

Chi-squared Distribution ${\chi_K}^2$



χ^2 and F Distributions

F Distribution

- If $U_1 \sim {\chi_{df_1}}^2$, $U_2 \sim {\chi_{df_2}}^2$ and U_1, U_2 are independent, then

$$X = \frac{U_1/df_1}{U_2/df_2} \sim F_{df_1,df_2}$$

- pdf of an F distributed random variable, X with df_1 and df_2 degrees of freedom is:

$$f(x) = \frac{\sqrt{\frac{(df_1\,x)^{df_1}\;df_2^{df_2}}{(df_1\,x+df_2)^{df_1+df_2}}}}{x\,\mathrm{B}\left(\frac{df_1}{2},\frac{df_2}{2}\right)}$$

  - $B(\cdot,\cdot)$ is the Beta function
- $E(X) = \frac{df_2}{df_2-2}$ for $df_2 > 0$

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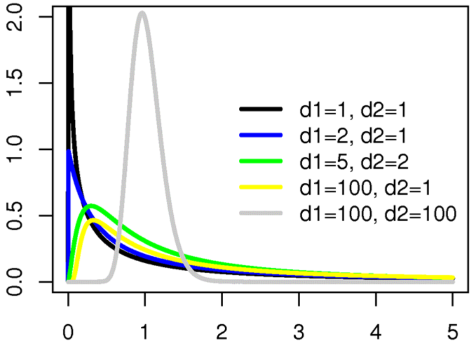
χ^2 and F Distributions

F-distribution



F Tests of fit

F-test of R^2

$$Y_i = \beta_0 + \beta_1 X_1 + \cdots + \beta_k X_k + u_i$$

$$H_0 : \beta_1 = \cdots = \beta_K = 0 \quad H_a : \text{at least one } \beta \neq 0$$

$$\frac{ESS/(K-1)}{RSS/(n-K)} = \frac{\frac{ESS}{TSS}/(K-1)}{\frac{RSS}{TSS}/(n-K)} = \frac{R^2/(K-1)}{(1-R^2)/(n-K)} \sim F(K-1, n-K)$$

Application

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F Tests of fit

Another application: incremental contribution of a set of variables

- $Y = \beta_1 + \beta_2 X_2 + u : \quad RSS_1$
- $Y = \beta_1 + \beta_2 X_2 + +\beta_3 X_3 + \beta_4 X_4 + u : \quad RSS_2$
- $H_0 : \beta_3 = \beta_4 = 0; \quad H_a : \beta_3 \neq 0$ or $\beta_4 \neq 0$ or both β_3 and $\beta_4 \neq 0$

$$\frac{\text{Increase in ESS}}{\text{cost in d.f.}} \Big/ \frac{\text{remaining RSS}}{\text{d.f. remaining}} \sim F(\text{cost, d.f. remaining})$$

$$\frac{(RSS_1 - RSS_2)/(df_1 - df_2)}{RSS_2/df_2} \sim F(df_1, df_2)$$

- Note: $F_{1,n}$ is the squared Student t_n distribution

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