MFin Econometrics I
Session 4: t-distribution, Simple Linear Regression, OLS assumptions and properties of OLS estimators

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t-distribution

What if σ_Y is un-known? (almost always)

- Recall: $\bar{Y} \sim N(\mu_Y, \frac{{\sigma_Y}^2}{n})$ and $\frac{\bar{Y}-\mu_Y}{\frac{\sigma_Y}{\sqrt{n}}} \sim N(0,1)$
    - If $(Y_1, ..., Y_n)$ are i.i.d. (and $0 < \sigma_Y^2 < \infty$) then, when n is large, the distribution of $\bar{Y}$ is well approximated by a normal distribution: Why? *CLT*
    - If $(Y_1, ..., Y_n)$ are independently and identically drawn from a Normal distribution, then *for any value of n*, the distribution of $\bar{Y}$ is normal: Why? *Sums of normal r.v.s are normal*
- But we almost never know σ_Y. We use sample standard deviation (s_Y) to estimate the unknown σ_Y
- Consequence of using the sample s.d. in the place of the population s.d. is an increase in uncertainty. *Why?*
    - $s_Y/\sqrt{n}$ is the standard error of the mean : estimate from sample, of st. dev. of sample mean, over all possible samples of size n drawn from the population

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t-distribution

Using s_Y: "estimator" of σ_Y

- ${s_Y}^2 = \frac{1}{\text{sample size-1}} \sum_{i=1}^{\text{sample size}} (Y_i - \bar{Y})^2$
  - Why sample size - 1 in this estimator?
  - Degrees of freedom (d.f.): the number of *independent observations* available for estimation. Sample size less the number of (linear) parameters estimated from the sample
- Use of an *estimator* (i.e., a random variable) for σ_Y (and thus, the use of an estimator for $\sigma_{\bar{Y}}$) motivates use of the fatter-tailed t-distribution in the place of $N(0,1)$
- The law of large numbers (LLN) applies to ${s_Y}^2$: ${s_Y}^2$ is, in fact, a "sample average" (How?)
- LLN: If sampling is such that $(Y_1, ..., Y_n)$ are i.i.d., (and if $E(Y^4) < \infty$), then ${s_Y}^2$ *converges in probability* to ${\sigma_Y}^2$:
  - ${s_Y}^2$ is an average, not of Y_i, but of its square (hence we need $E(Y^4) < \infty$)

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t-distribution

Aside: t-distribution probability density

$$f(t) = \frac{\Gamma(\frac{d.f.+1}{2})}{\sqrt{d.f.\pi}\Gamma(\frac{d.f.}{2})}(1 + \frac{t^2}{d.f.})^{-\frac{d.f.+1}{2}}$$

- $d.f.$: Degrees of freedom: the only parameter of the t-distribution;
  - $\Gamma()$ (Gamma function): $\Gamma(z) = \int_0^\infty t^{z-1}e^{-t}dt$
- Mean: $E(t) = 0$
- Variance: $V(t) = d.f./(d.f. - 2)$ for $d.f. > 2 \quad \geq 1$, converges to 1 as $d.f.$ increases (Compare: $N(0,1)$)

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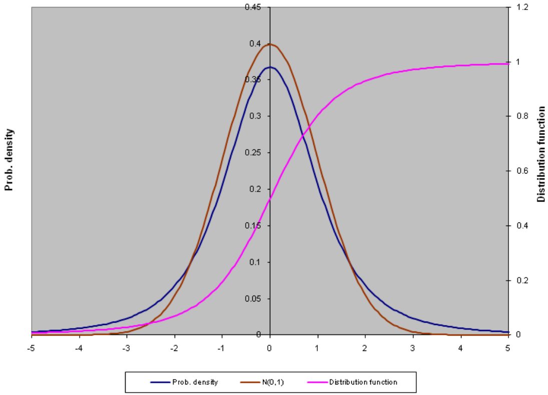
t-distribution

t-distribution pdf and CDF: d.f.=3



t-distribution

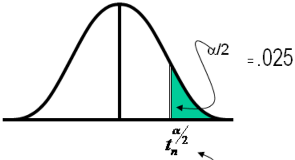
t - statistic

$$t_{d.f.} = \frac{\bar{Y} - \mu}{s_Y / \sqrt{n}} \qquad d.f. = n - 1$$

- test statistic for the sample average same form as z-statistic: Mean 0, Variance $\rightarrow 1$
- If the r.v. Y is is normally distributed in population, the test-statistic above for $\bar{Y}$ is t-distributed with $d.f. = n - 1$ degrees of freedom

t-distribution

t - distribution, table and use



Degrees of Freedom	$t_{.100}$	$t_{.05}$	$t_{.025}$	$t_{.01}$	$t_{.005}$
1	3.078	6.314	12.706	31.821	63.657
2	1.886	2.92	4.303	6.965	9.925
.	.	.	.	.	.
.	.	.	.	.	.
20	1.325	1.725	2.086	2.528	2.845
.	.	.	.	.	.
.	.	.	.	.	.
200	1.286	1.653	1.972	2.345	2.601
∞	1.282	1.645	1.96	2.326	2.576

t-distribution

$t \rightarrow$ SND

- If d.f. is moderate or large (> 30, say) differences between the t-distribution and $N(0,1)$ *critical values* are negligible
- Some 5% critical values for 2-sided tests:

degree of freedom	5%t-distribution critical value
10	2.23
20	2.09
30	2.04
60	2.00
∞	**1.96**

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t-distribution

Comments on t distribution, contd.

- The t distribution is only relevant when the sample size is small. But then, for the t distribution to be correct, the populn. distrib. of Y must be Normal Why?
  - The popln r.v. ($\bar{Y}$) must be Normally distributed for the test-statistic (with σ_Y estimated by s_Y) to be t-distributed
  - If sample size small, CLT does not apply; the popln distrib. of Y must be Normal for the popln. distrib. of $\bar{Y}$ to be Normal (sum of Normal r.v.s is Normal)
- So if sample size small and σ_Y estimated by s_Y, then test-statistic to be t-distributed if popln. is Normally distributed
- If sample size large (e.g., > 30), the popln. distrib. of $\bar{Y}$ is Normal irrespective of distrib. of Y - CLT (we saw that as d.f. increases, distrib. of $t_{d.f.}$ converges to $N(0, 1)$)
- Finance / Management data: Normality assumption dubious. e.g., earnings, firm sizes etc.

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t-distribution

Comments on t distribution, contd.

- So, with large samples,
- or, with small samples, but with σ known, and a normal population distribution

$$
\begin{aligned}
Pr(-z_{\alpha/2} \leq \frac{\bar{Y}-\mu}{\sigma/\sqrt{n}} \leq z_{\alpha/2}) &= 1-\alpha \\
Pr(\bar{Y} - z_{\alpha/2}\frac{\sigma}{\sqrt{n}} \leq \mu \leq \bar{Y} + z_{\alpha/2}\frac{\sigma}{\sqrt{n}}) &= 1-\alpha \\
Pr(\bar{Y} - 1.96\frac{\sigma}{\sqrt{n}} \leq \mu \leq \bar{Y} + 1.96\frac{\sigma}{\sqrt{n}}) &= 0.95
\end{aligned}
$$

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t-distribution

Comments on t distribution, contd.

- With small samples (<30 d.f.),
- drawn from an approximately normal population distribution,
- with the unknown σ estimated with s_Y,
- for test statistic $\frac{\bar{Y}-\mu}{s_Y/\sqrt{n}}$:

$$Pr(\bar{Y} - t_{d.f.,\alpha/2}\frac{s_Y}{\sqrt{n}} \leq \mu \leq \bar{Y} + t_{d.f.,\alpha/2}\frac{s_Y}{\sqrt{n}}) = 1 - \alpha$$

Q: What is $t_{d.f.,\alpha/2}$?

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Simple linear regression model

Population Linear Regression Model: Bivariate

$$Y = \beta_0 + \beta_1 X + u$$

- Interested in conditional probability distribution of Y given X
- Theory / Model : the conditional mean of Y given the value of X is a linear function of X
  - X : the independent variable or regressor
  - Y : the dependent variable
  - β_0 : intercept
  - β_1 : slope
  - u : regression disturbance, which consists of effects of factors other than X that influence Y, as also *measurement errors* in Y
  - n : sample size, $Y_i = \beta_0 + \beta_1 X_i + u_i \quad i = 1, \cdots, n$

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Simple linear regression model

Linear Regression Model - Issues

- Issues in estimation and inference for linear regression estimates are, at a general level, the same as the issues for the sample mean
- Regression coefficient is just a glorified mean
- Estimation questions:
  - How can we estimate our model? i.e., "draw" a line/plane/surface through the data ?
  - Advantages / disadvantages of different methods ?
- Hypothesis testing questions:
  - How do we test the hypothesis that the population parameters are *zero*?
    - Why test if they are zero?
  - How can we construct confidence intervals for the parameters?

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Simple linear regression model

Regression Coefficients of the Simple Linear Regression Model

- How can we estimate β_0 and β_1 from data?
  - $\bar{Y}$ is the ordinary least squares (OLS) estimator of μ_Y:
  - Can show that $\bar{Y}$ solves:

$$min_X \sum_{i=1}^{n} (Y_i - X)^2$$

- By analogy, OLS estimators of the unknown parameters β_0 and β_1, solve:

$$min_{\hat{\beta}_0,\, \hat{\beta}_1} \sum_{i=1}^{n} (Y_i - (\hat{\beta}_0 + \hat{\beta}_1 X_i))^2$$

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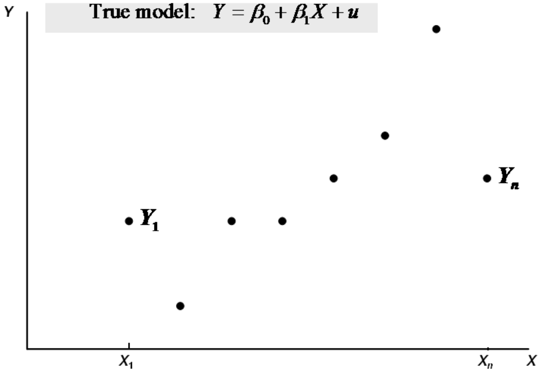
Simple linear regression model

OLS in pictures (1)



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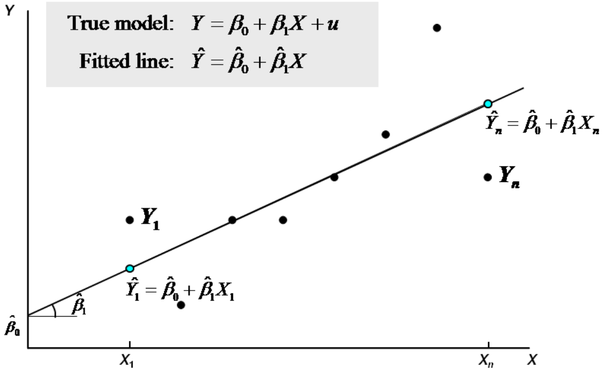
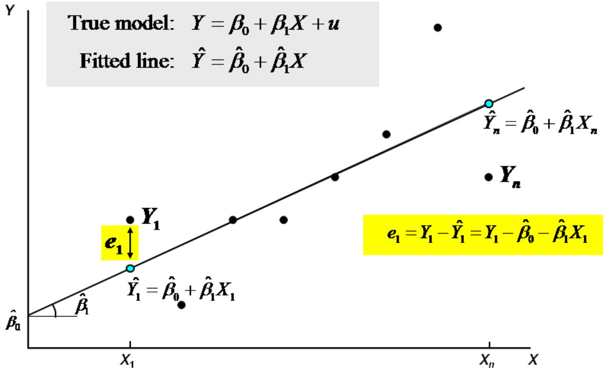
OLS in pictures (2)



Simple linear regression model

OLS in pictures (3)



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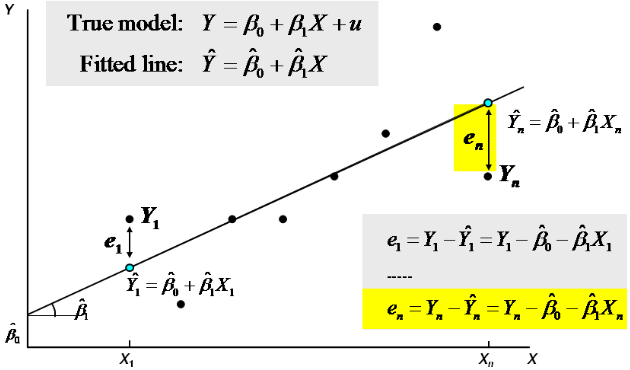
Simple linear regression model

OLS in pictures (4)



Simple linear regression model

OLS method of obtaining regression coefficients

- The sum : $e_1^2 + .. + e_n^2$, is the Residual Sum of Squares (RSS), a measure of total *error*
  - RSS is a function of both $\hat{\beta}_0$ and $\hat{\beta}_1$ (How?)

$$\begin{aligned} \text{RSS} &= (Y_1 - \hat{\beta}_0 - \hat{\beta}_1 X_1)^2 + .. + (Y_n - \hat{\beta}_0 - \hat{\beta}_1 X_n)^2 \\ &= \sum_{i=1}^{n} (Y_i - \hat{\beta}_0 - \hat{\beta}_1 X_i)^2 \end{aligned}$$

- Idea: Find values of $\hat{\beta}_0$ and $\hat{\beta}_1$ that minimise RSS
- $\frac{\partial RSS(\cdot)}{\partial \hat{\beta}_0} = -2 \sum (Y_i - \hat{\beta}_0 - \hat{\beta}_1 X_i) = 0$
- $\frac{\partial RSS(\cdot)}{\partial \hat{\beta}_1} = -2 \sum X_i (Y_i - \hat{\beta}_0 - \hat{\beta}_1 X_i) = 0$

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Simple linear regression model

Ordinary Least Squares, contd.

- $\frac{\partial RSS}{\partial \hat{\beta}_0} = 0$ and $\frac{\partial RSS}{\partial \hat{\beta}_1} = 0$
- Solving these two equations together:
- $\hat{\beta}_1 = \frac{\frac{1}{n}\sum(X_i - \bar{X})(Y_i - \bar{Y})}{\frac{1}{n}\sum(X_i - \bar{X})^2} = \frac{\sum(X_i - \bar{X})(Y_i - \bar{Y})}{\sum(X_i - \bar{X})^2} = \frac{Cov(X,Y)}{Var(X)}$
- $\hat{\beta}_0 = \bar{Y} - \hat{\beta}_1 \bar{X}$

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Simple linear regression model

Linear Regression Model: Interpretation

- Exercise

Linear regression model: Evaluation

Measures of Fit (i): Standard Error of the Regression (SER) and Root Mean Squared Error (RMSE)

- Standard error of the regression (SER) is an estimate of the dispersion (st.dev.) of the distribution of the disturbance term, u;
- Equivalently, of Y, conditional on X
- How close are Y values to $\hat{Y}$ values? can develop confidence intervals around any prediction
  - $SER = \sqrt{\frac{1}{n-2}\sum_{i=1}^{n}(e_i - \bar{e})^2} = \sqrt{\frac{1}{n-2}\sum_{i=1}^{n} {e_i}^2}$
  - SER converges to root mean squared error (RMSE)
  - $RMSE = \sqrt{\frac{1}{n}\sum_{i=1}^{n}(e_i - \bar{e})^2} = \sqrt{\frac{1}{n}\sum_{i=1}^{n} e_i^2}$
  - RMSE denominator has n SER has $(n-2)$
  - Why?

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Linear regression model: Evaluation

Measures of Fit (ii): R^2

- How much of the variance in Y can we explain with our model?
- Without the model, the best estimate of Y_i is the sample mean $\bar{Y}$
- With the model, the best estimate of Y_i is conditional on X_i and is the fitted value $\hat{Y}_i = \hat{\beta}_0 + \hat{\beta}_1 X_i$
- How much does the error in estimate of Y reduce with the model?

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Linear regression model: Evaluation

Goodness of fit

- The model: $Y_i = \hat{Y}_i + e_i$

$$\begin{aligned}
Var(Y) &= Var(\hat{Y} + e) \\
&= Var(\hat{Y}) + Var(e) + 2Cov(\hat{Y}, e) \\
&\overset{\text{Why?}}{=} Var(\hat{Y}) + Var(e) \\
\frac{1}{n}\sum(Y - \bar{Y})^2 &= \frac{1}{n}\sum(\hat{Y} - \bar{\hat{Y}})^2 + \frac{1}{n}\sum(e - \bar{e})^2 \\
\sum(Y - \bar{Y})^2 &= \sum(\hat{Y} - \bar{\hat{Y}})^2 + \sum(e - \bar{e})^2
\end{aligned}$$

- Total Sum of Squares (TSS) = Explained Sum of Squares (ESS) + Residual Sum of Squares (RSS)

$$R^2 = \frac{ESS}{TSS} = \frac{\sum(\hat{Y}_i - \bar{Y})^2}{\sum(Y_i - \bar{Y})^2}$$

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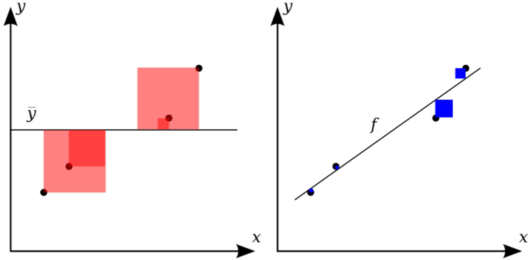
Linear regression model: Evaluation

Goodness of fit

- $TSS = ESS + RSS$
- $R^2 = \frac{ESS}{TSS} = \frac{\sum(\hat{Y}_i - \bar{Y})^2}{\sum(Y_i - \bar{Y})^2} = 1 - \frac{\sum e_i^2}{\sum(Y_i - \bar{Y})^2}$
- $\sqrt{R^2} = \frac{Cov(Y, \hat{Y})}{st.dev.(Y) st.dev.(\hat{Y})} = r_{Y,\hat{Y}}$



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Properties of OLS Estimators

After regression estimates

- In practical terms, we wish to:
  - quantify sampling uncertainty associated with $\hat{\beta}_1$
  - use $\hat{\beta}_1$ to test hypotheses such as $\beta_1 = 0$
  - construct confidence intervals for β_1
- all these require knowledge of the sampling distribution of the OLS estimators (based on the probability framework of regression)

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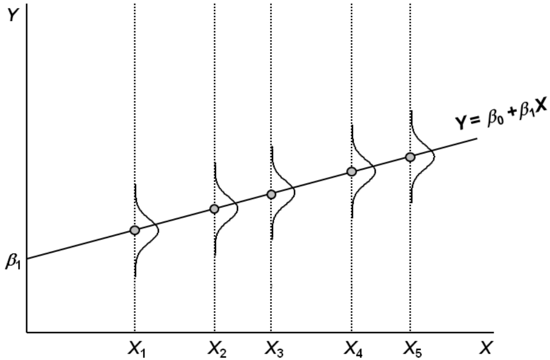
Properties of OLS Estimators

Properties of Estimators and the Least Squares Assumptions

- What kinds of estimators would we like?
- unbiased, efficient, consistent
- Under what conditions can these properties be guaranteed?
- We focus on the sampling distribution of $\hat{\beta}_1$ (Why not $\hat{\beta}_0$?)
  - The results below do hold for the sampling distribution of $\hat{\beta}_0$ too.

Properties of OLS Estimators

Assumption 1: $E(u|X = x) = 0$



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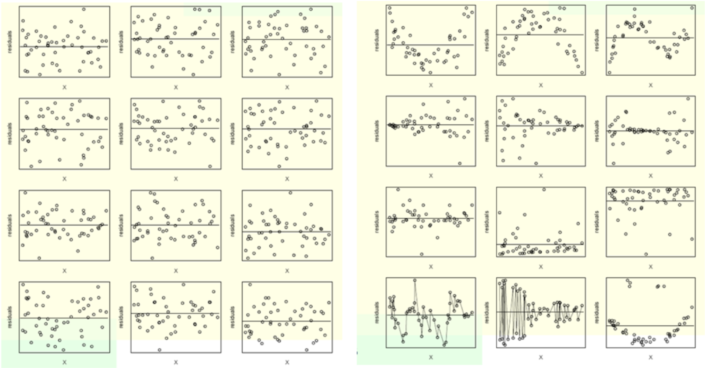
Properties of OLS Estimators

Assumption 1: $E(u|X = x) = 0$

- Conditional on X, u does not tend to influence Y either positively or negatively.
- Implication : Either X is not random, or,
- If X is random, it is distributed independently of the disturbance term, u: $Cov(X, u) = 0$
- This will be true if there are no relevant omitted variables in the regression model (i.e., those that are correlated to X)

Properties of OLS Estimators

Assumption 1: Residual plots that pass and that fail



Properties of OLS Estimators

Aside: include a constant in the regression

- Suppose $E(u_i) = \mu_u \neq 0$
- Suppose $u_i = \mu_u + v_i$ where $v_i \sim N(0, {\sigma_V}^2)$
- Then $Y_i = \beta_0 + \beta_1 X_i + v_i + \mu_u = (\beta_0 + \mu_u) + \beta_1 X_i + v_i$

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Properties of OLS Estimators

Assumption 2: (X_i, Y_i), $i = 1, ..., n$ are i.i.d.

- This arises naturally with simple random sampling procedure
- Because most estimators are linear functions of observations,
- Independence between observations helps in obtaining the sampling distributions of the estimators

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Properties of OLS Estimators

Assumption 3: Large outliers are rare

- A large outlier is an *extreme* value of X or Y
- Technically, $E(X^4) < \infty$ and $E(Y^4) < \infty$
  - Note: If X and Y are bounded, then they have finite fourth moments (income, etc.)
- Rationale : a large outlier can influence the results significantly

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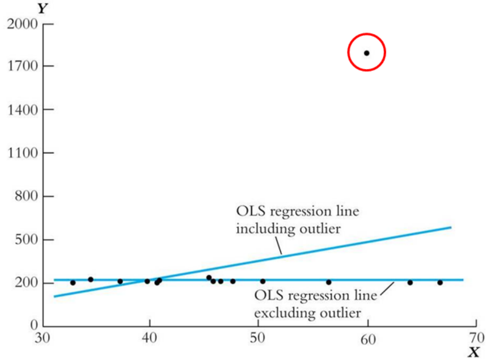
Properties of OLS Estimators

Assumption 3: OLS sensitive to outliers



Properties of OLS Estimators

Sampling distribution of $\hat{\beta}_1$

- If the three Least Squares Assumptions (mean zero disturbances, i.i.d. sampling, no large outliers) hold,
- then the exact (finite sample) sampling distribution of $\hat{\beta}_1$ is such that:
  - $\hat{\beta}_1$ is unbiased, that is, $E(\hat{\beta}_1) = \beta_1$
  - $Var(\hat{\beta}_1)$ can be determined
    - Other than its mean and variance, the exact distribution of $\hat{\beta}_1$ is complicated and depends on the distribution of (X, u)
  - $\hat{\beta}_1$ is consistent: $\hat{\beta}_1 \rightarrow_p \beta_1$ $\quad plim(\hat{\beta}_1) = \beta_1$
  - So when n is large, $\frac{\hat{\beta}_1 - \beta_1}{\sqrt{Var(\hat{\beta}_1)}} \sim N(0, 1)$ (by CLT)
- This parallels the sampling distribution of $\bar{Y}$

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Properties of OLS Estimators

Mean of the sampling distribution of $\hat{\beta}_1$

- $Y = \beta_0 + \beta_1 X + u$
- unbiasedness

$$\begin{aligned}
\hat{\beta}_1 &= \frac{Cov(X,Y)}{Var(X)} = \frac{Cov(X,[\beta_0 + \beta_1 X + u])}{Var(X)} \\
&= \frac{Cov(X,\beta_0) + Cov(X,\beta_1 X) + Cov(X,u)}{Var(X)} \\
&= \frac{0 + \beta_1 Cov(X,X) + Cov(X,u)}{Var(X)} \\
&= \beta_1 + \frac{Cov(X,u)}{Var(X)}
\end{aligned}$$

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Properties of OLS Estimators

Unbiasedness of $\hat{\beta}_1$

- $\hat{\beta}_1 = \frac{Cov(X,Y)}{Var(X)} = \beta_1 + \frac{Cov(X,u)}{Var(X)}$
- To investigate unbiasedness, take Expectation
- $E(\hat{\beta}_1) = \beta_1 + \frac{1}{Var(X)} E(Cov(X,u)) = \beta_1$
  - Expected value of $Cov(X,u)$ is zero (Why?)
- $\hat{\beta}_1$ is an unbiased estimator of β_1

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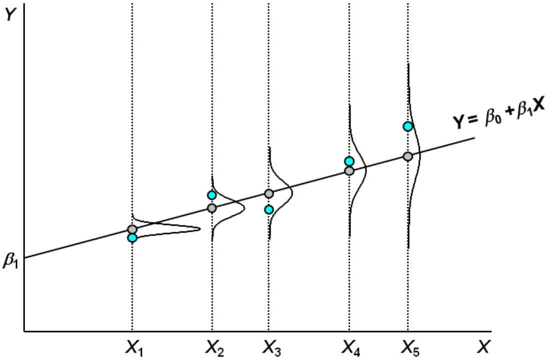
Linear regression model: further assumptions

Homoscedasticity and Normality of residuals

- u is homoscedastic
- $u \sim N(0, {\sigma_u}^2)$
- These assumptions are more restrictive
- However, if these assumptions are not violated, then other desirable properties obtain

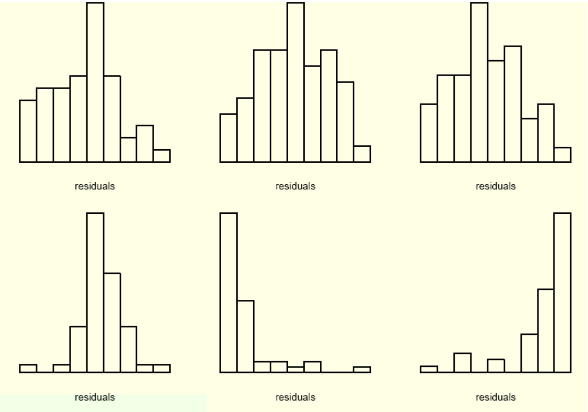
Linear regression model: further assumptions

Heteroscedasticity: one example



Linear regression model: further assumptions

Normality of disturbances: histograms of residuals that 'pass' and 'fail'



Linear regression model: further assumptions

Normality of disturbances 2: q-q plots of residuals that 'pass' and 'fail'